RAPID STABILIZATION IN A SEMIGROUP FRAMEWORK

AMBROISE VEST

ABSTRACT. We prove the well-posedness of a linear closed-loop system with an explicit (already known) feedback leading to arbitrarily large decay rates. We define a mild solution of the closed-loop problem using a dual equation and we prove that the original operator perturbed by the feedback is (up to the use of an extension) the infinitesimal generator of a strongly continuous group. We also give a justification to the exponential decay of the solutions. Our method is direct and avoids the use of optimal control theory.

1. Introduction

We consider a physical system which state x satisfies the Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \end{cases}$$

where A is a linear differential operator that models the dynamics of the system and B is a control operator that allows us to act on the system through a control u.

The *stabilization problem* consists in finding a *feedback operator* F such that the solutions of the *closed-loop problem*

$$x' = (A + BF)x$$

tend to zero as t tends to $+\infty$.

For finite-dimensional systems, D. L. Lukes [18] and D. L. Kleinman [10] (see also the book of D. L. Russell [20, pp. 112-117]) gave a systematic stabilization method thanks to an explicit feedback constructed with the controllability Gramian

$$\Lambda := \int_0^T e^{-tA} BB^* e^{-tA^*} dt.$$

The above matrix is positive-definite provided that (A, B) is exactly controllable (equivalently $(-A^*, B^*)$ is observable). In this case, the feedback

$$F := -B^* \Lambda^{-1}$$

stabilizes the system.

Later, adding a suitable weight-function inside the Gramian operator, M. Slemrod [21] adapted and improved this result to the case of infinite-dimensional systems with bounded control operators. More precisely, his feedback depends on a tuning

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parameter $\omega > 0$ that ensures a prescribed exponential decay rate of the solutions. The weighted Gramian

$$\Lambda_\omega := \int_0^T e^{-2\omega t} e^{-tA} BB^* e^{-tA^*} \, dt$$

is positive-definite if (A, B) is exactly controllable in time T (or $(-A^*, B^*)$ is exactly observable in time T). Then the solutions of the closed-loop problem provided with the feedback

$$F := -B^* \Lambda_\omega^{-1}$$

decrease to zero with an exponential decay rate being at least ω i.e. there is a positive constant c such that

$$\|x(t)\| \le ce^{-\omega t} \|x_0\|, \qquad t \ge 0,$$

for all initial data x_0, where $\| \cdot \|$ denotes a norm on the state space.

The problems that we have in mind are linear time-reversible partial differential equations (waves, plates...) with *boundary control*. These are infinite-dimensional problems and controlling only at (a part of) the boundary of the domain imposes that the control operator B is unbounded. This leads to difficulties in choosing the right functional spaces and the right notion of solution to have well-posed open-loop and closed-loop problems.

J.-L. Lions [17] gave an answer to the stabilization of such systems. His proof, using the theory of optimal control, is non-constructive and does not give any information on the decay rate of the solutions. By using a slightly different weight function in the above operator Λ_ω, V. Komornik [11] gave an explicit feedback leading to arbitrarily large decay rates. His approach does not use the theory of optimal control : an advantage is that one does not have to use strong existence and uniqueness results for infinite dimensional Riccati equations. In fact, the weight function is chosen in such a way that Λ_ω is the solution of an algebraic Riccati equation. Formally,

$$A\Lambda_\omega + \Lambda_\omega A^* + \Lambda_\omega C^* C \Lambda_\omega - BB^* = 0,$$

where a definition of the operator C and the rigourous meaning of this equation will be given later. For a presentation of this method of stabilization, see also the books of V. Komornik and P. Loreti [13, pp. 23-31] (where a generalization of this method to partial stabilization is also given) and J.-M. Coron [7, pp. 347-351].

Applications of this method to the boundary stabilization of the wave equation and the plates equation are given in [11]. This method can also be used to stabilize Maxwell equations [12] and elastodynamic systems [1]. Moreover, numerical and mechanical experiments ([4], [5], [22]) have proved the efficiency of this feedback.

In this paper, after recalling the construction of V. Komornik's feedback law and some results about the well-posedness of the open-loop problem (section 2), we give a proof of two points that were not justified in [11].

- The first point (section 3) is the well-posedness of the closed-loop problem with the explicit feedback introduced in [11]. Using the Riccati equation satisfied by Λ_ω, we introduce a "dual" closed-loop problem, which is easier to deal with because it does not involve the unbounded control operator B. Then we give a definition of *the mild solution* of the initial closed-loop problem and we prove in Theroem 3.1 that this solution satisfies a variation

of constants formula. To derive this formula, we adapt a representation formula of F. Flandoli [9] to the case of an algebraic Riccati equation. In Theorem 3.3, we prove that using a suitable extension $\widetilde{A}$ of A, the operator $\widetilde{A} - BB^*\Lambda_\omega^{-1}$ is the infinitesimal generator of a strongly continuous group on the original state space.

- The second point (section 4) consists in the justification of a formula, contained in Proposition 4.2, that is used in [11] to prove the exponential decay of the solutions. We recall at the end of the paper how this formula is used to obtain the exponential decay.

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2. A short review of the construction of the feedback and of the open-loop problem

2.1. **Hypotheses and notations.** The state space H and the control space U are Hilbert spaces. We denote by H' and U' their duals and by

$J : U' \to U$ the canonical isomorphism between U' and U;
$\widetilde{J} : H \to H'$ the canonical isomorphism between H and H'.

Moreover we make the following hypotheses :

- **(H1)** The operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous group e^{tA} on H. [1]
- **(H2)** $B \in L(U, D(A^*)')$, where $D(A^*)'$ denotes the dual [2] space of $D(A^*)$. Identifying $D(A^*)''$ with $D(A^*)$, we denote by $B^* \in L(D(A^*), U')$ the adjoint of B. This implies the existence of a number $\lambda \in \mathbb{C}$ and a bounded operator $E \in L(U, H)$ such that
$$B^* = E^*(A + \lambda I)^*.$$
- **(H3)** Given $T > 0$, there exists a constant $c_1(T) > 0$ such that
$$\int_0^T \|B^* e^{-tA^*} x\|_{U'}^2 dt \le c_1(T)\|x\|_{H'}^2$$
for all $x \in D(A^*)$. In the examples, this inequality represents a trace regularity result (see [14]). It is usually called the *direct inequality*.
- **(H4)** There exists a number $T > 0$ and a constant $c_2(T) > 0$ such that
$$c_2(T)\|x\|_{H'}^2 \le \int_0^T \|B^* e^{-tA^*} x\|_{U'}^2 dt$$
for all $x \in D(A^*)$. It is usually called the *inverse* or *observability inequality*.

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[1] Thus his adjoint $A^* : D(A^*) \subset H' \to H'$ is also the infinitesimal generator of a strongly continous group $e^{tA^*} = (e^{tA})^*$ on H'.

[2] Provided with the norm
$$\|x\|_{D(A^*)}^2 := \|x\|_{H'}^2 + \|A^* x\|_{H'}^2,$$
$D(A^*)$ is a Hilbert space. Moreover,
$$D(A^*) \subset H' \implies H \subset D(A^*)'.$$

Remark. Thanks to the assumptions (H1)-(H2), if the direct inequality in (H3) is satisfied for one $T > 0$, then it is satisfied for all $T > 0$. Moreover, the estimation remains true (up to a change of the constant in the right member) if we integrate on $(-T, T)$. Extending this inequality to all $x \in H'$ by density, the map $t \mapsto B^* e^{-tA^*} x$ can be seen as an element of $L^2_{\mathrm{loc}}(\mathbb{R}; U')$.

2.2. **Construction of the feedback.**

The operator Λ_ω. We suppose that the hypotheses (H1)-(H4) hold true (the number $T > 0$ giving the observability inequality in (H4)) and we recall the construction of the feedback exposed in [11], by defining a modified, weighted Gramian. We fix a number $\omega > 0$, set

$$T_\omega := T + \frac{1}{2\omega},$$

and we introduce a weight function on the interval $[0, T_\omega]$:

$$e_\omega(s) := \begin{cases} e^{-2\omega s} & \text{si} \quad 0 \le s \le T \\ 2\omega e^{-2\omega T}(T_\omega - s) & \text{si} \quad T \le s \le T_\omega. \end{cases}$$

Thanks to (H3) and (H4),

$$\langle \Lambda_\omega x, y \rangle_{H,H'} := \int_0^{T_\omega} e_\omega(s) \langle J B^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle_{U,U'} ds$$

defines a positive-definite self-adjoint operator $\Lambda_\omega \in L(H', H)$. Hence Λ_ω is invertible and we denote by $\Lambda_\omega^{-1} \in L(H, H')$ its inverse.

Actually the weight function e_ω has been chosen in such a way that the operator Λ_ω is solution to an algebraic Riccati equation. We are going to derive this Riccati equation because it will play a key role in the analysis of the well-posedness of the closed-loop problem and the exponential decay of the solutions.

An algebraic Riccati equation. Let $x, y \in D((A^*)^2)$. We compute the integral

$$\int_0^{T_\omega} \frac{d}{ds}\left[e_\omega(s) \langle J B^* e^{-sA^*} x, B^* e^{-sA^* y} \rangle_{U,U'} \right] ds \tag{1}$$

in two different ways. Note that the quantity between the brackets is differentiable in the variable s thanks to the regularity of x and y, and the hypothesis (H2) made on B^*.

- On the one hand, as $e_\omega(T_\omega) = 0$ and $e_\omega(0) = 1$, the above integral is

$$-\langle J B^* x, B^* y \rangle_{U,U'}.$$

- On the other hand, by differentiating inside the integral, we obtain

$$\int_0^{T_\omega} e'_\omega(s) \langle J B^* e^{-sA^*} x, B^* e^{-sA^*} y \rangle_{U,U'} ds - \int_0^{T_\omega} e_\omega(s) \langle J B^* e^{-sA^*} A^* x, B^* e^{-sA^*} y \rangle_{U,U'} ds$$
$$- \int_0^{T_\omega} e_\omega(s) \langle J B^* e^{-sA^*} x, B^* e^{-sA^*} A^* y \rangle_{U,U'} ds.$$

The formula

$$(Lx,y)_H := -\int_0^{T_\omega} e'_\omega(s)\langle JB^*e^{-sA^*}\Lambda_\omega^{-1}x, B^*e^{-sA^*}\Lambda_\omega^{-1}y\rangle_{U,U'}ds$$

defines a positive-definite self-adjoint operator $L \in L(H)$ because

$$\forall s \geq 0, \qquad -e'_\omega(s) \geq 2\omega e_\omega(s).$$

We set

$$C := \sqrt{L} \in L(H).$$

For $x, y \in H$, we have

$$\begin{aligned}(Lx,y)_H &= (Cx,Cy)_H \\ &= \langle Cx, \widetilde{J}Cy\rangle_{H,H'} \\ &= \langle x, C^*\widetilde{J}Cy\rangle_{H,H'}\end{aligned}$$

where $C^* \in L(H')$ is the adjoint of C. We can also remark the important [3] relation between C and Λ_ω^{-1} :

$$C^*\widetilde{J}C \geq 2\omega\Lambda_\omega^{-1}. \tag{2}$$

Finally the second computation of the integral gives

$$\begin{aligned}&-(L\Lambda_\omega x, \Lambda_\omega y)_H - \langle \Lambda_\omega A^*x, y\rangle_{H,H'} - \langle \Lambda_\omega x, A^*y\rangle_{H,H'} \\ &= -\langle C\Lambda_\omega x, \widetilde{J}C\Lambda_\omega y\rangle_{H,H'} - \langle \Lambda_\omega A^*x, y\rangle_{H,H'} - \langle \Lambda_\omega x, A^*y\rangle_{H,H'}.\end{aligned}$$

Putting together the two computations, we obtain the following algebraic Riccati equation satisfied by Λ_ω :

$$\begin{aligned}&\langle \Lambda_\omega A^*x, y\rangle_{H,H'} + \langle \Lambda_\omega x, A^*y\rangle_{H,H'} \\ &\qquad + \langle C\Lambda_\omega x, \widetilde{J}C\Lambda_\omega y\rangle_{H,H'} - \langle JB^*x, B^*y\rangle_{U,U'} = 0,\end{aligned} \tag{3}$$

first for $x, y \in D((A^*)^2)$ and then for $x, y \in D(A^*)$ by density of $D((A^*)^2)$ in $D(A^*)$ for the norm $\|\cdot\|_{D(A^*)}$.

An integral form of the algebraic Riccati equation. We rewrite the Riccati equation (3) in an integral form, verified for $x, y \in H$ instead of $x, y \in D(A^*)$. Set $x, y \in D(A^*)$. The equation (3) applied to $e^{-sA^*}x, e^{-sA^*}y \in D(A^*)$ gives

$$\begin{aligned}&\langle \Lambda_\omega A^*e^{-sA^*}x, e^{-sA^*}y\rangle_{H,H'} + \langle \Lambda_\omega e^{-sA^*}x, A^*e^{-sA^*}y\rangle_{H,H'} \\ &\quad + \langle C\Lambda_\omega e^{-sA^*}x, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle_{H,H'} - \langle JB^*e^{-sA^*}x, B^*e^{-sA^*}y\rangle_{U,U'} = 0.\end{aligned} \tag{4}$$

Integrating (4) bewteen 0 and t gives the following integral form of the Riccati equation (3) :

$$\begin{aligned}&\langle \Lambda_\omega x, y\rangle_{H,H'} = \langle \Lambda_\omega e^{-tA^*}x, e^{-tA^*}y\rangle_{H,H'} \\ &- \int_0^t \langle C\Lambda_\omega e^{-sA^*}x, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle_{H,H'}ds + \int_0^t \langle JB^*e^{-sA^*}x, B^*e^{-sA^*}y\rangle_{U,U'}ds.\end{aligned} \tag{5}$$

This relation remains true for $x, y \in H'$ by density of $D(A^*)$ in H' for the norm $\|\cdot\|_{H'}$.

[3] This estimation is important for the proof of the exponential decay of the solutions.

Rapid stabilization. Now let us recall the main result of [11].

Theorem 2.1 (Komornik, [11, p. 1597])**.** *Assume (H1)-(H4) for some $T > 0$. Fix $\omega > 0$ arbitrarily and set*

$$F := -JB^*\Lambda_\omega^{-1}.$$

Then the operator $A + BF$ generates a strongly continuous group [4] *in H and the solutions of the closed-loop problem*

$$x' = Ax + BFx, \qquad x(0) = x_0$$

satisfy the estimate [5]

$$\|x(t)\|_\omega \leq \|x_0\|_\omega e^{-\omega t}$$

for all $x_0 \in H$ and for all $t \geq 0$.

2.3. **Well-posedness of the open-loop problem.** In this paragraph, we recall some results about the well-posedness of the open-loop problem

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \tag{6}$$

where $u \in L^2_{\text{loc}}(\mathbb{R}; U)$. We would like to define a *mild solution* of this problem that is continuous and takes its values in H. The dificulty comes from the fact that the control operator is unbounded and takes its values in the larger space $D(A^*)'$. The next proposition will give an answer. [6]

Proposition 2.2 ([3, p. 259-260], [16, p. 648])**.** *Fix $T > 0$ and set*

$$z(t) := \int_0^t e^{(t-s)A} Eu(s)ds, \qquad -T \leq t \leq T.$$

Then

- *$z(t) \in D(A)$ for all $-T \leq t \leq T$;*
- *$\|(A + \lambda I)z(t)\|_H \leq k\|u\|_{L^2(-T,T;U)}$ for all $-T \leq t \leq T$, where $k > 0$ is a constant independant of u;*
- *$(A + \lambda I)z \in C([-T, T]; H)$.*

Definition. We define *the mild solution* of (6) as the application

$$x(t) = e^{tA}x_0 + (A + \lambda I)\int_0^t e^{(t-s)A} Eu(s)ds \tag{7}$$

which is continuous on $\mathbb{R}$ with values in H.

Remark. The relation (7) is a variation-of-constants-type formula. If B is bounded, this relation corresponds to

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} Bu(s)ds.$$

[4] As it was already noted in [11], we have to consider this affirmation in a weaker sense. More precisely, we will see that this is true if we replace A by a suitable extension.

[5] $\|\cdot\|_\omega$ defined by $\|x\|_\omega^2 := \langle \Lambda_\omega^{-1}x, x\rangle_{H',H}$, is a norm on H, equivalent to the usual norm thanks to the continuity and coercivity of Λ_ω^{-1}.

[6] This result is due to I. Lasiecka and R. Triggiani who first proved it in the case of hyperbolic equations with Dirichlet boundary conditions (see [14]).

Moreover we can also write the relation (7) by using the duality pairing :

$$\langle x(t), y\rangle_{H,H'} = \langle x_0, e^{tA^*}y\rangle_{H,H'} + \int_0^t \langle u(s), B^* e^{(t-s)A^*} y\rangle_{U,U'} ds,$$

for all $y \in D(A^*)$.

We end this section by recalling a regularity result. It concerns the solutions of the open-loop problem in the dual space H'

$$\begin{cases} y'(t) = -A^* y(t) + g(t), & t \in \mathbb{R}, \\ y(0) = y_0, \end{cases} \tag{8}$$

where $g \in L^1_{\text{loc}}(\mathbb{R}; H')$. This time, the source term does not involve any unbounded operator and the mild solution of (8) is defined by the "standard" variation of constants formula (see [19, p. 107]

$$y(t) = e^{-tA^*} y_0 + \int_0^t e^{-(t-r)A^*} g(r) dr, \tag{9}$$

which is a continuous function from $\mathbb{R}$ to H'. Thanks to the direct inequality stated in (H3), we can apply the operator B^* to the solution of the homogeneous problem associated to (8) (put $g = 0$ in (8)) and see this new function as an element of $L^2_{\text{loc}}(\mathbb{R}; U')$. Actually, this operation can be generalized to the solutions of the inhomogeneous problem (g can be $\neq 0$). We recall this result [7] in the

Proposition 2.3 ([9, pp. 92-93] , [16, p. 648])**.** *Fix $T > 0$. There exists a constant $c > 0$ such that for all $y_0 \in D(A^*)$ and all $g \in L^1(-T, T; D(A^*))$ we have the estimation*

$$\int_{-T}^{T} \|B^* y(t)\|^2_{U'} dt \le c\big(\|y_0\|^2_{H'} + \|g\|^2_{L^1(-T,T;U')}\big),$$

where y is defined by (9). By density, we can say that this estimation remains true for all initial data $y_0 \in H'$ and all source terms $g \in L^1(-T, T; H')$.

3. Well-posedness of the closed-loop problem

The aim of this section is to give a notion of solution to the closed-loop problem

$$\begin{cases} x'(t) = Ax(t) - BJB^*\Lambda_\omega^{-1} x(t), & t \in \mathbb{R}, \\ x(0) = x_0. \end{cases} \tag{10}$$

As for the open-problem (6), we can not use directly a variation of constants formula because of the unbounded perturbation $(-BJB^*\Lambda_\omega^{-1})$ of the infinitesimal generator A.

Let us give the main idea for the well-posedness of (10). The Riccati equation (3) can be rewritten *formally* as

$$A\Lambda_\omega + \Lambda_\omega A^* + \Lambda_\omega C^* \widetilde{J} C \Lambda_\omega - BJB^* = 0.$$

By multiplying the above equation on both side by Λ_ω^{-1}, we get

$$\Lambda_\omega^{-1} A + A^* \Lambda_\omega^{-1} + C^* \widetilde{J} C - \Lambda_\omega^{-1} BJB^* \Lambda_\omega^{-1} = 0. \tag{11}$$

[7] This result was firstly stated in [14] in the case of hyperbolic equations with Dirichlet boundary conditions.

Now we multiply the operator $A - BJB^*\Lambda_\omega^{-1}$ on the left by Λ_ω^{-1} and on the right by Λ_ω to get

$$\Lambda_\omega^{-1}(A - BJB^*\Lambda_\omega^{-1})\Lambda_\omega = \Lambda_\omega^{-1}A\Lambda_\omega - \Lambda_\omega^{-1}BJB^* = -A^* - C^*\widetilde{J}C\Lambda_\omega,$$

the last equality being a consequence of (11).

Remark. The two operators $A - BJB^*$ and $-A^* - C^*\widetilde{J}C\Lambda_\omega$ are (formally) *conjugated* by the operator Λ_ω.

The advantage of working with the conjugated operator is that the perturbation $(-C^*\widetilde{J}C\Lambda_\omega)$ is bounded. We are going to analyze the well-posedness of the closed-loop problem (10) by using the solutions of the "conjugated" closed-loop problem

$$\begin{cases} y'(t) = -A^*y(t) - C^*\widetilde{J}C\Lambda_\omega y(t), & t \in \mathbb{R}, \\ y(0) = y_0, \end{cases} \tag{12}$$

whose well-posedness is already known.

The perturbation being bounded, the operator $-A^* - C^*\widetilde{J}C\Lambda_\omega$, defined on $D(A^*)$ is the infinitesimal generator of a strongly continuous group $V(t)$ on H' (see [19, p. 22 and p. 76]). Moreover, for all $t \in \mathbb{R}$ and all $y_0 \in H'$ we have

$$V(t)y_0 = e^{-tA^*}y_0 - \int_0^t e^{-(t-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)y_0 dr. \tag{13}$$

Definition. Let $x_0 \in H$. We define *the mild solution* of (10) by

$$U(t)x_0 := \Lambda_\omega V(t)\Lambda_\omega^{-1}x_0.$$

Now we prove that this notion of solution is "coherent" with the closed-loop problem (10) in the sense that it satisfies a variation of constants formula, close to the one that we would formally use.

Theorem 3.1. *$U(t)$ is a strongly continuous group in H whose generator is*

$$A_U := \Lambda_\omega(-A^* - C^*\widetilde{J}C\Lambda_\omega)\Lambda_\omega^{-1}; \qquad D(A_U) = \Lambda_\omega D(A^*).$$

Moreover, it satisfies the variation of constants formula

$$\langle U(t)x_0, y\rangle = \langle e^{tA}x_0, y\rangle - \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}y\rangle dr, \tag{14}$$

for all $x_0 \in H$ and $y \in H'$.

Remark. The formula (14) does not mean that $A - BJB^*\Lambda_\omega^{-1}$ is the infinitesimal generator of a group (or even a semigroup) but it justifies the choice of $U(t)$ to define the mild solution of the closed-loop problem (10). To justify that the integral in (14) is meaningful, see the remark after the Lemma just below.

Remark. Formula (14) can be rewritten as

$$U(t)x_0 = e^{tA}x_0 - (A + \lambda I)\int_0^t e^{(t-r)A}EJB^*\Lambda_\omega^{-1}U(r)x_0 dr, \tag{15}$$

for all $x_0 \in H$. We can show (15) first for $x_0 \in D(A^*)$ and extend it by density.

The proof of Theorem 3.1 relies on the following representation formula of Λ_ω.

Lemma 3.2. *Set $x, y \in H'$ and $t \in \mathbb{R}$. Then*

$$\langle \Lambda_\omega x, y\rangle_{H,H'} = \langle \Lambda_\omega V(t)x, e^{-tA^*}y\rangle_{H,H'} + \int_0^t \langle JB^*V(s)x, B^*e^{-sA^*}y\rangle_{U,U'}ds. \tag{16}$$

Remark. The integral in the above formula is meaningful. Indeed the first part of the bracket defines an element of $L^2_{\text{loc}}(\mathbb{R}; U)$ because of (13) and the extended regularity result stated in Proposition 2.3. The second part of the bracket defines an element of $L^2_{\text{loc}}(\mathbb{R}; U')$ thanks to the direct inequality stated in (H3).

Proof of Theorem 3.1. At first, $U(t)$ is a C_0-group on H because it is the conjugate group (by Λ_ω) of $V(t)$. The relation between the infinitesimal generator of $V(t)$ and those of $U(t)$ is also a general fact about conjugate semigroups (see [8, p. 43 and p. 59]).

To prove relation (14), we use relation (16) in which we replace x by $\Lambda_\omega^{-1}x_0$ and y by $e^{tA^*}y$. Finally we use the definition of $U(t)$, that is $U(t)x_0 = \Lambda_\omega V(t)\Lambda_\omega^{-1}x_0$. $\square$

Proof of Lemma 3.2. F. Flandoli has proved in [9] a similar relation for the solution of a differential Riccati equation. We adapt his proof to the case of an algebraic Riccati equation. The proof contains two steps : at first, we use the integral form of the Riccati equation (5) and the variation of constants formula for V (13) to prove relation (16) modulo a rest. Then we show that this rest vanishes. [8]

First step. Fix $x, y \in H'$ and $t \in \mathbb{R}$. From (5) and (13) we have

$$\begin{aligned}
&\langle \Lambda_\omega x, y\rangle \\
=&\langle \Lambda_\omega \left[e^{-tA^*}x\right], e^{-tA^*}y\rangle - \int_0^t \langle C\Lambda_\omega \left[e^{-sA^*}x\right], \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle ds \\
&+ \int_0^t \langle JB^*\left[e^{-sA^*}x\right], B^*e^{-sA^*}y\rangle ds \\
=&\langle \Lambda_\omega \left[V(t) + \int_0^t e^{-(t-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)dr\right]x, e^{-tA^*}y\rangle \\
&- \int_0^t \langle C\Lambda_\omega \left[V(s) + \int_0^s e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)dr\right]x, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle ds \\
&+ \int_0^t \langle JB^*\left[V(s) + \int_0^s e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)dr\right]x, B^*e^{-sA^*}y\rangle ds \\
=&\langle \Lambda_\omega V(t)x, e^{-tA^*}y\rangle + \int_0^t \langle JB^*V(s)x, B^*e^{-sA^*}y\rangle ds + R.
\end{aligned}$$

Second step. To obtain relation (16), we have to show that the rest R vanishes. To lighten the writing, we set

$$g(r) := C^*\widetilde{J}C\Lambda_\omega V(r)x \in C(\mathbb{R}; H').$$

[8] In order to simplify the notations, we will omit the name of the spaces under the duality brackets in this proof.

Let us rewrite the rest :

$$\begin{aligned}
R =&\langle \Lambda_\omega \int_0^t e^{-(t-r)A^*} g(r)dr, e^{-tA^*}y\rangle \\
&- \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle ds \\
&- \int_0^t \langle C\Lambda_\omega \int_0^s e^{-(s-r)A^*} g(r)dr, \widetilde{J}C\Lambda_\omega e^{-sA^*}y\rangle ds \\
&+ \int_0^t \langle JB^* \int_0^s e^{-(s-r)A^*} g(r)dr, B^* e^{-sA^*}y\rangle ds. \\
=:& R_1 - R_2 - R_3 + R_4.
\end{aligned}$$

- We can also write R_1 as

$$R_1 = \int_0^t \langle \Lambda_\omega e^{-(t-r)A^*} g(r), e^{-(t-r)A^*} e^{-rA^*}y\rangle dr.$$

The integrand of the above integral corresponds to the first term in the right member of (5) by replacing x by $C^*\widetilde{J}C\Lambda_\omega V(r)x = g(r)$, y by $e^{-rA^*}y$ and t by $t-r$. Hence

$$\begin{aligned}
R_1 =& \int_0^t \langle \Lambda_\omega g(r), e^{-rA^*}y\rangle dr \\
&+ \int_0^t \Big[\int_0^{t-r} \langle C\Lambda_\omega e^{-sA^*} g(r), \widetilde{J}C\Lambda_\omega e^{-sA^*} e^{-rA^*}y\rangle ds \Big] dr \\
&- \int_0^t \Big[\int_0^{t-r} \langle JB^* e^{-sA^*} g(r), B^* e^{-sA^*} e^{-rA^*}y\rangle ds \Big] dr \\
=:& R_1' + R_2' - R_3'.
\end{aligned}$$

- We have

$$R_1' = R_2.$$

The change of variable $\sigma := s + r$ and Fubini's theorem give

$$\begin{aligned}
R_2' =& \int_0^t \int_r^t \langle C\Lambda_\omega e^{-(\sigma-r)A^*} g(r), \widetilde{J}C\Lambda_\omega e^{-\sigma A^*}y\rangle d\sigma dr \\
=& \int_0^t \int_0^\sigma \langle C\Lambda_\omega e^{-(\sigma-r)A^*} g(r), \widetilde{J}C\Lambda_\omega e^{-\sigma A^*}y\rangle dr d\sigma \\
=& R_3.
\end{aligned}$$

- Il remains to show that $R_3' = R_4$. Difficulties arise since the operator B^* is unbounded. The idea is to construct two approximations $R_3'(n)$ and $R_4(n)$ for R_3' and R_4. We show that $R_3'(n) = R_4(n)$ and that $R_3'(n)$ and $R_4(n)$ converge respectively to R_3' and R_4.

Remark. A^* is the infinitesimal generator of a C_0-group in H'. Hence for sufficiently large $n \in \mathbb{N}$, n lies in the resolvant set of A^*. We set

$$I_n := n(nI - A^*)^{-1} \in L(H').$$

Then for all $x \in H'$, $I_n x \in D(A^*)$ and $I_n x \to x$ as $n \to \infty$ (see [19, Lemma 3.2. p. 9]). Moreover, the sequence $\|I_n\|$ is bounded from above independently of n.

Indeed, as A^* is the generator of a group, it results from Hille-Yosida theorem ([19, Theorem 6.3 p. 23]) that for sufficiently large $n \in \mathbb{N}$,

$$\|I_n\| = \|n(nI - A^*)^{-1}\| \leq \frac{n\alpha}{n-\beta},$$

where α and β are two positive constants.

- For n sufficiently large, we set

$$R_3'(n) := \int_0^t \Big[\int_0^{t-r} \langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y \rangle ds \Big] dr.$$

The application between the duality bracket is measurable on the product space $(0,t) \times (0,t)$. [9] Moreover

$$\begin{aligned}
&\int_0^t \int_0^{t-r} \Big| \langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y \rangle \Big| ds dr \\
\leq &\int_0^t \int_0^t \Big| \langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y \rangle \Big| ds dr \\
= &\int_0^t \Big[\int_0^t \Big| \langle JB^* e^{-sA^*} I_n g(r), B^* e^{-sA^*} e^{-rA^*} y \rangle \Big| ds \Big] dr \qquad \text{(Fubini-Tonelli)} \\
\leq &c \int_0^t \|g(r)\|_{H'} \|e^{-rA^*}\|_{H'} dr \qquad \text{(Cauchy-Schwarz, direct inequality)} \\
< &\infty.
\end{aligned}$$

Hence we can invert the order of the integrals in $R_3'(n)$. We get (first by doing the change of variable $\sigma := s + r$) :

$$\begin{aligned}
R_3'(n) &= \int_0^t \int_r^t \langle JB^* e^{-(\sigma - r)A^*} I_n g(r), B^* e^{-\sigma A^*} y \rangle d\sigma dr \\
&= \int_0^t \int_0^\sigma \langle JB^* e^{-(\sigma - r)A^*} I_n g(r), B^* e^{-\sigma A^*} y \rangle dr d\sigma.
\end{aligned}$$

Finally, $R_3'(n) = \int_0^t \varphi_n(r) dr$ et $R_3' = \int_0^t \varphi(r) dr$ with the evident notations. For all $0 \leq r \leq t$, we have

$$\begin{aligned}
|\varphi_n(r) - \varphi(r)| =& \Big| \int_0^{t-r} \langle JB^* e^{-sA^*} [I_n g(r) - g(r)], B^* e^{-sA^*} e^{-rA^*} y \rangle ds \Big| \\
\leq& \int_0^t \Big| \langle JB^* e^{-sA^*} [I_n g(r) - g(r)], B^* e^{-sA^*} e^{-rA^*} y \rangle \Big| ds \\
\leq& c \|I_n g(r) - g(r)\|_{H'} \|e^{-rA^*} y\|_{H'} \quad \text{(Cauchy-Schwarz and direct inequality).}
\end{aligned}$$

Hence $\varphi_n(r) \to \varphi(r)$ as $n \to \infty$. Thanks to Cauchy-Schwarz, the direct inequality and because $\|I_n\|$ is bounded from above, we have

$$|\varphi_n(r)| \leq c \|I_n g(r)\|_{H'} \|e^{-rA^*} y\|_{H'} \leq c' \|g(r)\|_{H'} \|e^{-rA^*} y\|_{H'}.$$

We can apply the dominated convergence theorem : $R_3'(n) \to R_3$.

[9] The right side is measurable because it is the composition of two measurable functions. (we recall that $B^* e^{-tA^*}$ is well-defined in $L^2_{\text{loc}}(\mathbb{R}; U')$). In the left side we can replace B^* by $B_k^* := E^*(A_k^* + \bar{\lambda} I)$ where $A_k^* \in L(H')$ is the *Yosida approximation* of A^* (see [19]). For all $x \in D(A^*)$, $B_k^* x \to B^* x$ as $k \to \infty$ and $B_k^* \in L(H', U')$. Hence, the left-hand side of the duality bracket is measurable as a simple limit of continuous (hence measurable) functions on $(0,t) \times (0,t)$.

- For sufficiently large n, we set

$$R_4(n) := \int_0^t \langle JB^* \int_0^s e^{-(s-r)A^*} I_n g(r) dr, B^* e^{-sA^*} y \rangle ds.$$

But I_n et $e^{-(s-r)}A^*$ commute and

$$\begin{aligned} B^* I_n &= E^*(A + \lambda I)^* n(nI - A^*)^{-1} \\ &= -nE^* + (n^2 + n\lambda) E^*(nI - A^*)^{-1} \in L(H'). \end{aligned}$$

Hence (see [2, p. 139] for interverting B^* and the integral)

$$B^* \int_0^s e^{-(s-r)A^*} I_n g(r) dr = \int_0^s B^* I_n e^{-(s-r)A^*} g(r) dr$$

and

$$R_4(n) = \int_0^t \int_0^s \langle JB^* I_n e^{-(s-r)A^*} g(r), B^* e^{-sA^*} y \rangle dr ds = R_3'(n).$$

Finally, for all $0 \le r \le t$, $I_n g(r) \to g(r)$ and $\|I_n g(r)\| \le c\|g(r)\|$, the right member being integrable on $(0, t)$. Thanks to the dominated convergence theorem, $I_n g \to g$ in $L^1(0, t; H')$. The estimation of proposition 2.3 gives

$$B^* \int_0^s e^{-(s-r)A^*} I_n g(r) dr \to B^* \int_0^s e^{-(s-r)A^*} g(r) dr$$

in $L^2(0, t; U')$. Hence $R_4(n) \to R_4$ and by unicity of the limit, $R_3' = R_4$. □

We can do a little better and link the infinitesimal generator of $U(t)$ to the original operators involved in the closed-loop problem (10).

Theorem 3.3. *The operator $A : D(A) \subset H \to H$ can be extended to a bounded operator $\widetilde{A} : H \to D(A^*)$. The operator*

$$\widetilde{A} - BJB^* \Lambda_\omega^{-1}$$

coincides with A_U (the generator of $U(t)$) on $D(A_U) = \Lambda_\omega D(A^*)$ i.e.

$$\forall x_0 \in D(A_U), \qquad (\widetilde{A} - BJB^* \Lambda_\omega^{-1}) x_0 = A_U x_0 \in H.$$

We first recall a classical extension result for the unbounded operator A to a bounded operator on H with values on the extrapolation space $D(A^*)'$ (see [15, pp. 6-7] and [6, pp. 21-22]).

Lemma 3.4. *The operator $A : D(A) \subset H \to H$ admits a unique extension to an operator $\widetilde{A} \in L(H, D(A^*)')$. Moreover this extension satisfies the relation*

$$\langle \widetilde{A} x, y \rangle_{D(A^*)', D(A^*)} = \langle x, A^* y \rangle_{H, H'}. \tag{17}$$

for all $x \in H$ and $y \in D(A^*)$.

Proof of Lemma 3.4. The unicity of such an extension is the consequence of the density of $D(A)$ in H. Provided with the norm $\|\cdot\|_{D(A^*)}$, $D(A^*)$ is a Hilbert space and $A^* \in L(D(A^*), H')$. We denote by $\widetilde{A}$ the (Banach-)adjoint of A^* seen as a bounded operator between the Banach spaces $D(A^*)$ and H. Hence the definition of the (Banach-)adjoint gives

$$\widetilde{A} \in L(H, D(A^*)')$$

and for all $x \in H$ and $y \in D(A^*)$,

$$\langle \widetilde{A}x, y\rangle_{D(A^*)', D(A^*)} = \langle x, A^* y\rangle_{H,H'}$$

i.e. relation (17) is true. Moreover this new operator $\widetilde{A}$ defines extension of A i.e. the two operators coincides on $D(A)$. Indeed from the above relation specialized to $x \in D(A) \subset H$, we get

$$\forall y \in D(A^*), \quad \langle \widetilde{A}x, y\rangle_{D(A^*)', D(A^*)} = \langle Ax, y\rangle_{H,H'} \quad \Rightarrow \quad Ax = \widetilde{A}x \in H. \qquad \square$$

Proof of Theorem 3.3. Instead of returning to the Riccati equation (3), we are going to differentiate the variation of constants formula (14). We know that for $x_0 \in \Lambda_\omega D(A^*) = D(A_U)$, the map

$$t \mapsto U(t)x_0$$

is differentiable and

$$\frac{d}{dt}U(t)x_0 = A_U U(t)x_0.$$

In particular if $y \in H'$, then [10]

$$\langle \frac{d}{dt}U(t)x_0, y\rangle = \langle A_U U(t)x_0, y\rangle.$$

Differentiating (14) with respect to t, we want to link the generator A_U and the operator $A - BJB^*\Lambda_\omega^{-1}$ (a priori with values in $D(A^*)'$). We remark that defining the domain of the latter operator is not clear. Let $x_0 \in \Lambda_\omega D(A^*)$ and $y \in D((A^*)^2)$.

First step. The map

$$r \mapsto B^*\Lambda_\omega^{-1}U(r)x_0$$

is continuous from $\mathbb{R}$ to U'. Indeed, setting $y_0 := \Lambda_\omega^{-1}x_0 \in D(A^*)$, we have

$$\begin{aligned}
B^*\Lambda_\omega^{-1}U(r)x_0 &= B^*\Lambda_\omega^{-1}\Lambda_\omega V(r)\Lambda_\omega^{-1}x_0 \\
&= B^*V(r)y_0 \\
&= E^*(A^* + \bar{\lambda}I)V(r)y_0 \\
&= E^*(A^* + C^*\widetilde{J}C\Lambda_\omega - C^*\widetilde{J}C\Lambda_\omega + \bar{\lambda}I)V(r)y_0 \\
&= -E^*(-A^* - C^*\widetilde{J}C\Lambda_\omega)V(r)y_0 + E^*(-C^*\widetilde{J}C\Lambda_\omega + \bar{\lambda}I)V(r)y_0 \\
&= -E^*V(r)(-A^* - C^*\widetilde{J}C\Lambda_\omega)y_0 + E^*(-C^*\widetilde{J}C\Lambda_\omega + \bar{\lambda}I)V(r)y_0,
\end{aligned}$$

the latter expression being continuous in r.

Remark. On $D(A^*) = D(-A^* - C^*\widetilde{J}C\Lambda_\omega)$, the operators $V(r)$ et $-A^* - C^*\widetilde{J}C\Lambda_\omega$ (generator of $V(r)$) commute (this is a general fact about semigroups) but a priori $V(r)$ and A^* do not commute.

Second step. The map

$$s \mapsto B^*e^{sA^*}y$$

is differentiable on $\mathbb{R}$ with values in U'. Indeed, as $y \in D((A^*)^2)$, we have $(A^* + \bar{\lambda}I)y \in D(A^*)$ and

$$B^*e^{sA^*}y = E^*e^{sA^*}(A^* + \bar{\lambda}I)y.$$

The latter expression is differentiable with respect to s and its derivative is $B^*e^{sA^*}A^*y$.

[10] Again, when the name of spaces under the duality brackets are unnecessary, we omit them.

Third step. We deduce from the two previous steps that the map

$$t \mapsto \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}y\rangle dr$$

is differentiable on $\mathbb{R}$ and its derivative is the map

$$t \mapsto \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}A^*y\rangle dr + \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle.$$

It results that given two (regular) data $x_0 \in \Lambda_\omega D(A^*)$ and $y \in D((A^*)^2)$, we can differentiate $\langle U(t)x_0, y\rangle$ with respect to t and get

$$\begin{aligned}\frac{d}{dt}\langle U(t)x_0, y\rangle =& \langle e^{At}x_0, A^*y\rangle - \int_0^t \langle JB^*\Lambda_\omega^{-1}U(r)x_0, B^*e^{(t-r)A^*}A^*y\rangle dr\\ &- \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle.\end{aligned}$$

Replacing y by A^*y in (14) and reinjecting in the above relation, we obtain

$$\frac{d}{dt}\langle U(t)x_0, y\rangle = \langle U(t)x_0, A^*y\rangle - \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle. \tag{18}$$

With the same regularity as above for x_0 et y, we have

$$\frac{d}{dt}\langle U(t)x_0, y\rangle_{H,H'} = \langle A_U U(t)x_0, y\rangle_{H,H'} = \langle A_U U(t)x_0, y\rangle_{D(A^*)',D(A^*)},$$

where A_U is the infinitesimal generator of $U(t)$. We recall from Lemma 3.4 that A admits a unique extension to an aperator $\widetilde{A} \in L(H, D(A^*)')$. Thanks to this extension we can link A_U and $A - BJB^*\Lambda_\omega^{-1}$. From (18) and (17) we have, for $x_0 \in \Lambda_\omega D(A^*)$ and $y \in D((A^*)^2)$,

$$\begin{aligned}\frac{d}{dt}\langle U(t)x_0, y\rangle_{H,H'} &= \langle U(t)x_0, A^*y\rangle_{H,H'} - \langle JB^*\Lambda_\omega^{-1}U(t)x_0, B^*y\rangle_{U,U'}\\ &= \langle \widetilde{A}U(t)x_0, y\rangle_{D(A^*)',D(A^*)} - \langle BJB^*\Lambda_\omega^{-1}U(t)x_0, y\rangle_{D(A^*)',D(A^*)}\\ &= \langle (\widetilde{A} - BJB^*\Lambda_\omega^{-1})U(t)x_0, y\rangle_{D(A^*)',D(A^*)}\\ &= \langle A_U U(t)x_0, y\rangle_{D(A^*)',D(A^*)}\end{aligned}$$

In particular, the latter equality is true for $t = 0$. Hence, given a fixed $x_0 \in \Lambda_\omega D(A^*)$, we have

$$\langle (\widetilde{A} - BJB^*\Lambda_\omega^{-1})x_0, y\rangle_{D(A^*)',D(A^*)} = \langle A_U x_0, y\rangle_{D(A^*)',D(A^*)},$$

for all $y \in D((A^*)^2)$. This relation remains true for all $y \in D(A^*)$ by density of $D((A^*)^2)$ in $D(A^*)$ (for the norm $\|\cdot\|_{D(A^*)}$). Finally,

$$\forall x_0 \in \Lambda_\omega D(A^*) = D(A_U), \qquad (\widetilde{A} - BJB^*\Lambda_\omega^{-1})x_0 = A_U x_0 \in H. \qquad \square$$

Remark. With an unbounded control operator (i.e. $B \in L(U, D(A^*)'))$, one can prove, through examples, that the domain of A_U is not always included in the domain of A. Thus, in general, the extension $\widetilde{A}$ is necessary in order to link A_U and A on $D(A_U)$ (i.e. we cannot omit the "tilde" in the above relation). This phenomenon does not appear with a bounded control operator (i.e. $B \in L(U, H)$): in that case, we can prove that the spaces $D(A)$ and $D(A_U)$ coincide.

4. Derivation of a representation formula for Λ_ω^{-1} and exponential decay

In this section, we give a justification to a representation formula for Λ_ω^{-1} involving the group $U(t)$. This corresponds to the formula (3.11) in [11]. We recall it as it is written in [11] : for all $s, t \in \mathbb{R}$,

$$\Lambda_\omega^{-1} = U(t-s)^* \Lambda_\omega^{-1} U(t-s) + \int_s^t U(\tau - s)^* (C^* \widetilde{J} C + \Lambda_\omega^{-1} B J B^* \Lambda_\omega^{-1}) U(\tau - s) d\tau.$$

This formula is used in [11] to prove the exponential decay of the solutions of the closed-loop system. Again, F. Flandoli derived an analog formula in the case of differential Riccati equations in [9]. We adapt his proof to the case of algebraic Riccati equations.

We first prove a similar representation formula for Λ_ω.

Proposition 4.1. *For all $x, y \in H'$ and $t \in \mathbb{R}$*

$$\langle \Lambda_\omega x, y \rangle_{H,H'} = \langle \Lambda_\omega V(t) x, V(t) y \rangle_{H,H'} + \int_0^t \langle J B^* V(s) x, B^* V(s) y \rangle_{U,U'} ds + \int_0^t \langle C \Lambda_\omega V(s) x, \widetilde{J} C \Lambda_\omega V(s) y \rangle_{H,H'} ds. \tag{19}$$

Proof. It relies on the representation formula (16) for Λ_ω that we have already proved : for $x, y \in H'$,

$$\langle \Lambda_\omega x, y \rangle = \langle \Lambda_\omega V(t) x, [e^{-tA^*} y] \rangle + \int_0^t \langle J B^* V(s) x, B^* [e^{-sA^*} y] \rangle ds.$$

In the right member of the above relation, we replace $e^{-tA^*} y$ and $e^{-sA^*} y$ by using the variation of constants formula (13) for V :

$$\begin{aligned}
\langle \Lambda_\omega x, y \rangle =& \langle \Lambda_\omega V(t) x, V(t) y \rangle \\
&+ \langle \Lambda_\omega V(t) x, \int_0^t e^{-(t-s)A^*} C^* \widetilde{J} C \Lambda_\omega V(s) y ds \rangle \\
&+ \int_0^t \langle J B^* V(s) x, B^* V(s) y \rangle ds \\
&+ \int_0^t \langle J B^* V(s) x, B^* \int_0^s e^{-(s-r)A^*} C^* \widetilde{J} C \Lambda_\omega V(r) y dr \rangle ds \\
=:& T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

But

$$T_2 := \int_0^t \langle \Lambda_\omega V(t-s) V(s) x, e^{-(t-s)A^*} C^* \widetilde{J} C \Lambda_\omega V(s) y \rangle ds.$$

Thanks to (16), applied to $V(s)x$ instead of x, $C^*C\Lambda_\omega V(s)y$ instead of y and $t-s$ instead of t, we have

$$\begin{aligned}
T_2 = &\int_0^t \langle \Lambda_\omega V(s)x, C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle ds \\
&- \int_0^t \int_0^{t-s} \langle JB^*V(r)V(s)x, B^*e^{-rA^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle drds \\
= &\int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle ds \\
&- \int_0^t \int_0^{t-s} \langle JB^*V(r+s)x, B^*e^{-rA^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle drds.
\end{aligned}$$

The change of variable $\sigma := r+s$ in the last term gives

$$\begin{aligned}
T_2 = &\int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle ds \\
&- \int_0^t \int_s^t \langle JB^*V(\sigma)x, B^*e^{-(\sigma-s)A^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle d\sigma ds \\
= &\int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle ds \\
&- \int_0^t \int_0^\sigma \langle JB^*V(\sigma)x, B^*e^{-(\sigma-s)A^*}C^*\widetilde{J}C\Lambda_\omega V(s)y\rangle dsd\sigma.
\end{aligned}$$

Hence we have proved that

$$\begin{aligned}
\langle \Lambda_\omega x, y\rangle = &\langle \Lambda_\omega V(t)x, V(t)y\rangle \\
&+ \int_0^t \langle JB^*V(s)x, B^*V(s)y\rangle ds \\
&+ \int_0^t \langle C\Lambda_\omega V(s)x, \widetilde{J}C\Lambda_\omega V(s)y\rangle ds \\
&+ \int_0^t \langle JB^*V(s)x, B^* \int_0^s e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)ydr\rangle ds \\
&- \int_0^t \int_0^s \langle JB^*V(s)x, B^*e^{-(s-r)A^*}C^*\widetilde{J}C\Lambda_\omega V(r)y\rangle drds.
\end{aligned}$$

We have already shown in the proof of Lemma 3.2 that the two last terms in the above relation cancel each other. Hence the relation is proved. □

Proposition 4.2. *For all $x, y \in H$ and $t \in \mathbb{R}$*

$$\langle \Lambda_\omega^{-1}x, y\rangle_{H',H} = \langle \Lambda_\omega^{-1}U(t)x, U(t)y\rangle_{H',H} \tag{20}$$
$$+ \int_0^t \langle \widetilde{J}CU(s)x, CU(s)y\rangle_{H',H} ds + \int_0^t \langle JB^*\Lambda_\omega^{-1}U(s)x, B^*\Lambda_\omega^{-1}U(s)y\rangle_{U,U'} ds.$$

Proof. We replace x by $\Lambda_\omega^{-1}x$ and y by $\Lambda_\omega^{-1}y$ in the relation given by the Proposition 4.1 :

$$\begin{aligned}\langle x, \Lambda_\omega^{-1}y\rangle =& \langle \Lambda_\omega V(t)\Lambda_\omega^{-1}x, V(t)\Lambda_\omega^{-1}y\rangle + \int_0^t \langle JB^*V(s)\Lambda_\omega^{-1}x, B^*V(s)\Lambda_\omega^{-1}y\rangle ds \\ &+ \int_0^t \langle \widetilde{J}C\Lambda_\omega V(s)\Lambda_\omega^{-1}x, C\Lambda_\omega V(s)\Lambda_\omega^{-1}y\rangle ds.\end{aligned}$$

Then, by definition of U,

$$\begin{aligned}\langle \Lambda_\omega^{-1}x, y\rangle &= \langle \Lambda_\omega^{-1}U(t)x, U(t)y\rangle \\ &+ \int_0^t \langle JB^*\Lambda_\omega^{-1}U(s)x, B^*\Lambda_\omega^{-1}U(s)y\rangle ds + \int_0^t \langle \widetilde{J}CU(s)x, CU(s)y\rangle ds.\end{aligned}$$ □

Remark. A simple change of variable implies that for all $s, t \in \mathbb{R}$ and all $x, y \in H$,

$$\begin{aligned}\langle \Lambda_\omega^{-1}x, y\rangle &= \langle \Lambda_\omega^{-1}U(t-s)x, U(t-s)y\rangle \\ &+ \int_s^t \langle JB^*\Lambda_\omega^{-1}U(\tau-s)x, B^*\Lambda_\omega^{-1}U(\tau-s)y\rangle d\tau + \int_s^t \langle \widetilde{J}CU(\tau-s)x, CU(\tau-s)y\rangle d\tau.\end{aligned} \tag{21}$$

Finally, let us recall the outline of the proof of the exponential decay of the solutions of the closed-loop problem (10). We denote by $x(t)$ the mild solution of (10) i.e.

$$x(t) = U(t)x_0.$$

Using the relation (21) with $x = y = U(s)x_0 = x(s)$, we have

$$\begin{aligned}\langle \Lambda_\omega^{-1}x(s), x(s)\rangle &= \langle \Lambda_\omega^{-1}x(t), x(t)\rangle \\ &+ \int_s^t \langle JB^*\Lambda_\omega^{-1}x(\tau), B^*\Lambda_\omega^{-1}x(\tau)\rangle d\tau + \int_s^t \langle \widetilde{J}Cx(\tau), Cx(\tau)\rangle d\tau.\end{aligned}$$

Let $0 \le s \le t$. The estimation (2) between C and Λ_ω^{-1} and the positiveness of the second term of the right member in the above relation yield

$$\langle \Lambda_\omega^{-1}x(s), x(s)\rangle \ge \langle \Lambda_\omega^{-1}x(t), x(t)\rangle + 2\omega \int_s^t \langle \Lambda_\omega^{-1}x(\tau), x(\tau)\rangle d\tau.$$

A Gronwall-type lemma (see [11, p. 1599]) gives

$$\|x(t)\|_\omega^2 \le \|x_0\|_\omega^2 e^{-2\omega t} \qquad \forall t \ge 0.$$

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Institut de Recherche Mathématique Avancée, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg Cédex, France

E-mail address: ambroise.vest@math.unistra.fr